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Braided Logic: The Simplest Models[1]

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The only pair of binary classical essential connectives for which the Artin braid relation holds is a distributive lattice, i.e. the Boolean disjunction and conjunction of

1. EQUIVALENT CATEGORIES

An equational logic, a language with axioms, was invented essentially by Birkhoff (1935). The best short survey is given by Tarski (1968). An equational logic whose axioms include the Artin braid relation (Artin, 1926, 1947) is said to be *a braided logic*.

A language is a functor from a category of alphabets to a category of free operads (clones, algebras of terms, operads of words, etc). In this paper a language is generated from the alphabets of 0-, 1-, and 2-cells, given by directed graphs. A collection G_0 of 0-cells is a free monoid generated by singleton 0-cell $1 = |$, then $2 = \|$, etc, and therefore there is a bijection $G_0 \simeq \mathbb{N}$. A model of a 1-cell $\in G_1$ is a functor or a concrete operation. A model of a 2-cell $\in G_2$ is a natural transformation of functors. An alphabet of 1-cells $\{\alpha, \beta, \ldots\} \subset G_1$ is generated by grafting a free operad G_1 (alias a clone) of derived (grafted) 1-cells; see, e.g., the tree operad in Graczyńska

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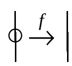
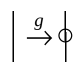
and Oziewicz (1999). An alphabet of 2-cells $\subset G_2$ (alias the abstract operations), like $\{f \in \text{nat}(\alpha, \beta), \ldots\}$, expands to 2-cells among grafted 1-cells (in a model, expands to natural transformations of grafted derived functors). A collection of all expanded 2-cells is a partial algebra with respect to partial compositions.

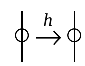
For example, let an alphabet of 1-cells consist of two letters α and β together with a 2-cell $f \in \text{nat}(\alpha, \beta)$. Then a 2-cell f is said to be of *type* (α, β), i.e., a type in this case is a pair of ordered 1-cells. A model of an abstract operation $f \in \text{nat}(\alpha, \beta)$ is a gebra of a type (α, β), (α, β)-gebra, i.e., a pair (A, fA), where A is an object and the 1-morphism fA: $\alpha A \to \beta A$ is a βA-valued concrete operation on an object A. A family of all gebras of a given type $\{(A, fA), f \in \text{nat}(\alpha, \beta)\}$ is an (α, β)-category.

Let $\text{id} \in G_1$ be some identity 1-cell. Then an (α, id)-gebra is an algebra of a type α, and an (id, β)-gebra is said to be a cogebra of a type β. In what follows, we use the following endofunctors:

$\tau A \equiv A \times A,$ $\qquad$ $\alpha A \equiv A \times S \times A,$ $\qquad$ $\gamma A \equiv (A^A) \times (A^A)$







(τ, id)-gebra $\qquad$ (id, τ)-gebra (co-gebra) $\qquad$ (τ, τ)-gebra

A pair of a binary concrete operations on a set A, say $\wedge A$: $A^{\times 2} \to A$ and $\vee A$: $A^{\times 2} \to A$, is the same as one ternary operation $f\,A$: $A \times S \times A \to A$, where S is a set of two 'sorts' $S \equiv \{\wedge, \vee\}$. Equivalently, one can consider the $A^{\times 2}$-valued binary operation hA: $A^{\times 2} \to A^{\times 2}$. These concrete operations must be considered as the models of an abstract algebra $f \in \text{nat}(\alpha, \text{id})$ and 'braided' gebra $h \in \text{nat}(\tau, \tau)$,

The following diagram illustrates an example where a model for $g \in \text{nat}(\text{id}, \gamma)$ is a concrete 'cooperation' gA: $A \to (A^A) \times (A^A)$:

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ & & \downarrow{\scriptstyle gA} \\ A & \xrightarrow{\gamma} & (A^A) \times (A^A) \end{array}$$

The categories of (α, id)-gebras, (id, γ)-gebras, and (τ, τ)-gebras are equivalent. The aim of this paper is to study equational classes of 'braided' gebras in a category of type (τ, τ) as a possible alternative for the classical equational classes of type (α, id), for example, a distributive lattice, i.e., the Boolean binary classical connectives disjunction and conjunction.

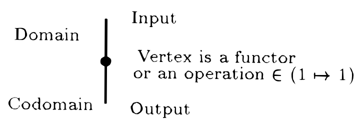
We found that on a two-element set there are exactly 43 models of the Artin braid axiom (see Fig. 3) for $h \in \text{nat}(\tau, \tau)$ among *a priori* 256 possible operations of type $(2 \mapsto 2)$.

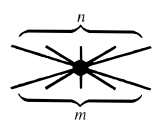
2. FREE OPERAD OF DIRECTED GRAPHS

All graphs here are directed from the top to the bottom and arrows are omitted. We assume that a collection of 0-cells G_0 is a free monoid on one generator $|$, therefore G_0 is isomorphic to monoid of natural numbers $(\mathbb{N}, +)$. A one-dimensional edge corresponds to generating a 0-cell (therefore an edge do *not* need to be incident to vertices), and 1-cells (always incident to pair of 0-cells) correspond to (zero-dimensional) vertices. This convention is the same as in, e.g., Yetter (1990), Joyal and Street (1991), Lyubashenko (1995), and Graczyńska and Oziewicz (1999); we have

geometry		algebra
$1 \overset{\text{dim}}{\longleftarrow} \vert$	$\longleftrightarrow$	category object $\overset{\text{grade}}{\longrightarrow} 0$
$0 \overset{\text{dim}}{\longleftarrow} \bullet$	$\longleftrightarrow$	functor $\overset{\text{grade}}{\longrightarrow} 1$

In what follows, $T^n_m \equiv (n \mapsto m) \subset G_1$ denotes a subset of all 1-cells of a fixed type (n, m),

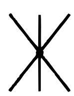




$\in T^n_m$ or $(n \mapsto m)$

A model of a 0-cell $|$ is a category or an object of a category. A model of a 1-cell can be a functor or a concrete operation,

$\longmapsto$ $\mathrm{id}_{\mathbf{cat}}$ or id_A


$$\longmapsto \begin{array}{lll} f(D,E,F) & \in (A \times B \times C)^{(D\times E\times F)} & \text{(functor)} \\ fA & \in (A \times A \times A)^{(A\times A\times A)} & \text{(operation)} \end{array}$$

An operad of graphs is a collection of graphs $\langle n \mapsto m\rangle$ closed with respect to multivalued *grafting*. This is illustrated by example on Fig. 1.

In Fig. 1, the first row gives some indivisible letters, one-letter words. In the next two rows, there are examples of two-letter grafted words. In the second row, the first two 1-cells comes from a concatenation of 'indivisible' 1-cells, and this is the same as 0-*grafting*. Therefore 0-grafting is nothing more than the monoidal structure of 0-cells. The last two 1-cells in the second row are 1-grafted from the first and third indivisible letters. The open circles in the last row on Fig. 1 show where the 1-*grafting* takes place.

If a monoidal structure on 0-cells is modeled by Cartesian product $\times$ of sets, then a 1-cell of type $(n \mapsto 2)$ in this realization is a map $A^{\times n} \mapsto A^{\times 2}$.

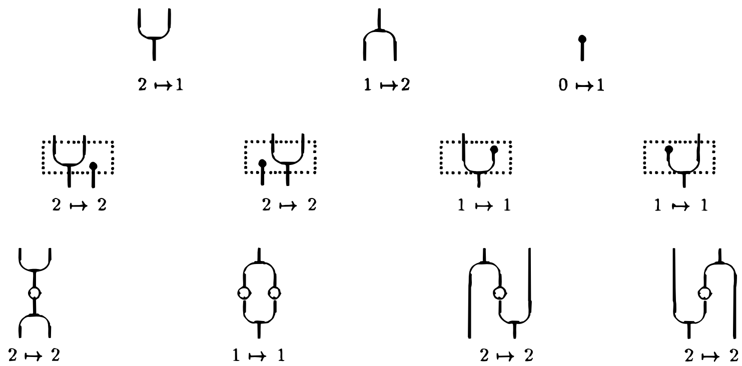


Fig. 1. The first row gives an example of an alphabet of 1-cells (by definition, 'indivisible' letters). The next rows gives grafted graphs.

In this case, every such 1-cell is no longer indivisible and is the same as a pair of n-ary operations $\langle\alpha, \beta\rangle$ of a type $(n \mapsto 1)$. In what follows, we assume that

$$T^n_m = \underbrace{T^n \times \cdots \times T^n}_{m \text{ times}}, \qquad T^n \equiv T^n_1$$

In particular, a 1-cell of type $1 \mapsto 2$ is equivalent to a pair of unary 1-cells of type $1 \mapsto 1$. A 1-cell of type $1 \mapsto 2$ is said to be a *duplication* if the following two identities hold:

A mitosis of cells in biology is an example of a duplication $1 \mapsto 2$.

A 1-cell from $(2 \mapsto 2)$ is said to be a *crossing*. We are assuming that every crossing 1-cell is not indivisible and is the same as a pair of binary operations $\langle\alpha, \beta\rangle$ of type $(2 \mapsto 1)$. Therefore a crossing 1-cell of a type $(2 \mapsto 2)$ is the same as a brassiere $\langle\alpha, \beta\rangle$ of the binary operations (Fig. 2).

2.1. Grading: *k*-Essential Operations

Definition 1. An n-ary 1-cell $f \in T^n$ is said to be k-essential if f cannot be grafted from any collection of m-ary 1-cells for $m < k$, including a killer $\in (1 \mapsto 0)$. A set of all n-ary k-essential 1-cells is denoted by E^n_k, $k \leq n$.

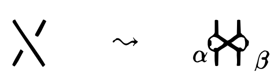
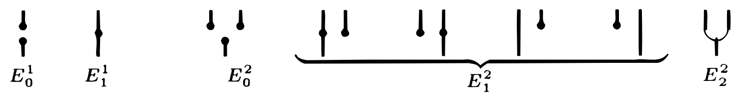


Fig. 2. A crossing 1-cell can be equivalent to a brassiere.

A set of all n-ary 1-cells T^n is a union of the disjoint sets of k-essential 1-cells for $0 \le k \le n$,

$$T^n = E^n_n \cup E^n_{n-1} \cup \ldots \cup E^n_0, \qquad E^n_i \cap E^n_j = \emptyset \qquad \text{if} \quad i \neq j$$



On two-elements sets, $|E^1_1| = 2$, $|E^1_0| = 2$, $|E^2_2| = 10$, $|E^2_1| = 4$, $|E^2_0| = 2$.

3. THE ARTIN BRAID AXIOM

A braided logic includes the Artin braid axiom shown on Fig. 3 (Artin, 1926, 1947). The Artin braid relation on grafted 1-cells of type ($3 \mapsto 3$) (in the first row in Fig. 3), after inserting a brassiere from Fig. 2, is equivalent to the three ternary relations shown in the second row in Fig. 3. If $\sigma \equiv (\alpha, \beta) \in (2 \mapsto 2)$, then the Artin braid equation is given by a hexagon,

$$(\text{id} \times \sigma) \circ (\sigma \times \text{id}) \circ (\text{id} \times \sigma) = (\sigma \times \text{id}) \circ (\text{id} \times \sigma) \circ (\sigma \times \text{id})$$

A solution of the Artin relation in a realization on sets is said to be a braid. In our convention, a braid does not need to be invertible. Comparing Fig. 3 with the above equation, we see the advantage of the graphical approach: graphs convey more information.

A set of all derivable identities (an equivalence binary relation on 1-cells $\subset G_1 \times G_1$) follows from identities given in Fig. 3 by means of the five Birkhoff rules of direct inference, including the tautologies and substitutions (Birkhoff, 1935; Tarski, 1968).

4. RESULTS

We found all models of the Artin relation as shown in Fig. 3 on two-element carriers. On the two-element set $|A| = 2$, there are 16 binary operations, as shown in Table I, and there are $4^4 = 256$ operations of type ($2 \mapsto 2$).

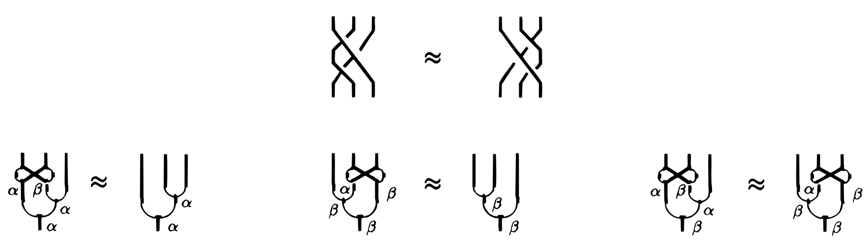


Fig. 3. The Artin axiom.

Table I. Binary Operations in Two-Valued Logic

2-Essential	$\vee$ Disjunction	
	$\equiv$ Equivalence or biconditional	
	$\wedge$ Conjunction	Associative
	$\not\equiv$ Inequivalence	
	$\subset$ Reciprocal implication	
	$\supset$ Conditional or implication	
	$\|$ Anticonjunction or Sheffer's stroke	Nonassociative
	$\not\supset$ Negation of implication	
	$\not\subset$ Negation of reciprocal implication	
	$\downarrow$ Antidisjunction or Pierce's arrow	
1-Essential	p Proposition p	Associative
	q Proposition q	
	$\sim p$ Negation of p	Nonassociative
	$\sim q$ Negation of q	
0-Essential	Always true	Associative
	Always false	

Theorem 2. There are 43 models of the Artin braid relation on the two-element set. Braids in which at least one binary operation is 2-essential are presented in Table III and all models are classified in Table IV.

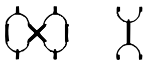
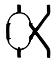
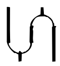
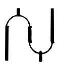
Proof. PC symbolic programs are available to the Artin relation. An independent symbolic package appropriate for logic has been elaborated by Makaruk (2000). Note that in the tables, type = essentiality. ■

One important corollary is that the only pair of binary distinct 2-essential connectives (see Table I) for which the Artin braid relation holds is a distributive lattice, i.e., the Boolean disjunction and conjunction. Therefore a distributive lattice of a type $\langle 2, 2\rangle$ can be equivalently defined by the Artin braid

Table II. Binary 2-Essential Operations E_2^2

Input		Disjunction	Pierce's $\downarrow$	$\supset$	$\not\supset$	Conjunction
T	T	T	F	T	F	T
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	F	F
		Sheffer's $\|$	Equivalence	Inequivalence	$\subset$	$\not\subset$
T	T	F	T	F	T	F
T	F	T	F	T	T	F
F	T	T	F	T	F	T
F	F	T	T	F	T	F

Table III. Braids in Which at Least One Binary Operation is 2-Essential[a]

	Type	Number of models	Braids
	⟨2, 2⟩	4	Conjunction, disjunction
	⟨2, 1⟩	11	Conjunction, disjunction, implication, negation of reciprocal implication
			Conjunction, disjunction
	⟨1, 2⟩		Conjunction, disjunction, reciprocal implication
			Conjunction, disjunction
	⟨2, 0⟩ ⟨0, 2⟩	8	Conjunction, disjunction, equivalence, inequivalence

[a] All are idempotents, $\sigma^2 = \sigma$.

identity given in Fig. 3 with the additional condition that $\alpha \neq \beta$ and $\alpha, \beta \in E_2^2$.

There are only three binary 2-essential operations from E_2^2 which do not contribute to Artin braids: Pierce's arrow (antidisjunction), Sheffer's stroke (anticon junction), and negation of the implication.

For $\sigma \in T_2^2$ and $m \in \mathbb{N}$, let $\sigma^m \equiv \sigma \circ \sigma \ldots \in T_2^2$ and $\sigma^0 \equiv \mathrm{id}$.

Definition 3. The relation $\sigma^n = \sigma^m$ for $m < n$ *is* said to be the minimal relation if for all $0 \leq k < m$ and $0 \leq k < l < n$, $\sigma^k \neq \sigma^l$.

Table IV. The Number of Models with Respect to a Type and with Respect to a Minimal Relation

	$\sigma = \mathrm{id}$	$\sigma^2 = \mathrm{id}$	$\sigma^2 = \sigma$	$\sigma^3 = \sigma$	$\sigma^3 = \sigma^2$	$\sigma^4 = \mathrm{id}$
⟨2, 2⟩			4			
⟨2, 1⟩ and ⟨1, 2⟩			4	3	4	
⟨2, 0⟩ and ⟨0, 2⟩			8			
⟨1, 1⟩	1	2	4	1		2
⟨0, 1⟩ and ⟨1, 0⟩			3	1	4	
⟨0, 0⟩			2			

Proposition 4. Let among α and β from $(2 \mapsto 1)$ given in Tables I and II, at least one binary operation be 2-essential. Then a braid $\sigma \equiv (\alpha, \beta)$ is idempotent, $\sigma^2 = \sigma$.

In Table III, the 11 braids of essentiality $\langle 2, 1\rangle$ and $\langle 1, 2\rangle$ come from the binary 1-essential identity operations grafted with the killer.

In Table III, all braids from $E_2^2 \times E_2^2$ (of an essentiality $\langle 2, 2\rangle$) are idempotents. They are combinations of *conjunction* and *disjunction*. This gives the following result:

Corollary 5. Disjunction and conjunction are braid-symmetrical, i.e., for $\sigma \equiv (\wedge, \vee)$, $\wedge \circ \sigma = \wedge$ and $\vee \circ \sigma = \vee$.

5. CONCLUSION AND SOME DIRECTIONS FOR FURTHER STUDIES

On two-element sets, we found all models of an equational theory whose axiom is given by the Artin relation in Fig. 3.

If, moroever, we ask that a braid is a not diagonal element from $E_2^2 \times E_2^2$, i.e., that binary operations must be 2-essential and not the same, than there is only one model of the Artin relation, that is, conjunction and disjunction. This means that the classical distributive lattice can be equivalently described in terms of the Artin relation for a pair of not the same binary 2-essential operations.

In this paper, we restricted analysis to the Artin relation only. However, the braided logic also must include morphism identities,

This deserves further study.

A generalization of the Artin hexagon to braided higher categories is the Zamolodchikov tetrahedron as the coherence condition (e.g., Kapranov and Voevodsky, 1994; Fischer, 1994; Baez and Dolan, 1995; Baez and Neuchl, 1996). We belive that it is worth determining the simplest nontrivial models of an equational logic for braided higher categories with the Zamolodchikov tetrahedron equation as the basic axiom. To explain human intelligence and the layers in the biological structure of the brain, most probably we would need braided higher categories.

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